The Augmented Complex Gaussian Kernel LMS Algorithm

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Outline

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This procedure is equivalent with a non linear processing in F.

Reproducing Kernel Hilbert Spaces.

Consider a linear class $\mathcal{H}$ of real (complex) valued functions f defined on a set X (in particular $\mathcal{H}$ is a Hilbert space), for which there exists a function $\kappa : \mathcal{X} \times \mathcal{X} \to \mathbb{R}(\mathbb{C})$ with the following two properties:

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1. For every $x \in \mathcal{X}$, $\kappa(x, \cdot)$ belongs to $\mathcal{H}$.
2. κ has the so called reproducing property, i.e.,

$$f(x) = \langle f, \kappa(x, \cdot)\rangle_{\mathcal{H}}, \text{ for all } f \in \mathcal{H}, \ x \in \mathcal{X}. \tag{1}$$

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$\mathcal{X} \ni x \to \Phi(x) := \kappa(x, \cdot) \in \mathcal{H}$
$\mathcal{X} \ni y \to \Phi(y) := \kappa(y, \cdot) \in \mathcal{H}$,
then the inner product in $\mathcal{H}$ is given as a function computed on $\mathcal{X}$:

$$\boxed{\kappa(x, y) = \langle \kappa(x, \cdot), \kappa(y, \cdot) \rangle_{\mathcal{H}}} \quad \text{kernel trick}$$

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- Different types of nonlinearities can be treated in a unified way.

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  - Develop the Algorithm in $\mathcal{X}$.
  - Express it, **if possible**, in inner products.
  - Replace inner products with kernel evaluations according to the kernel trick.
- Work directly in the RKHS, assuming that the data have been mapped and live in the RKHS $\mathcal{H}$, i.e.,

$$\mathcal{X} \ni \boldsymbol{x} \to \Phi(\boldsymbol{x}) := \kappa(\boldsymbol{x}, \cdot) \in \mathcal{H}.$$

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- The complex domain not only provides a convenient and elegant representation for these signals, but also a natural way to preserve their characteristics and to handle transformations that need to be performed.
- In the more traditional setting, one usually assumes that the signal is *circular*.

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- Naturally this assumption limits the area for applications, as many practical signals exhibit non-circular characteristics.
- Widely linear filters are able to efficiently treat such signals, as they capture the full second order statistics of any given complex-valued data sequence.

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- Moreover, in the related literature the complex kernel functions have been ignored.
- Recently, however, a unified kernel-based framework, which is able to treat complex signals, has been presented.
- This machinery transforms the input data into complex RKHS and employs the Wirtinger's Calculus to derive the respective gradients.

Feature Map

- In the case of the pure complex kernels, we map directly the data to the complex RKHS using the corresponding feature map, i.e.,

$$\boldsymbol{\Phi}(\boldsymbol{z}) = \kappa_{\mathbb{C}}(\cdot, \boldsymbol{z}).$$

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   - Reproducing Kernel Hilbert Spaces
   - Complex valued signals
   - Complex RKHS
   - The Problem
2. Widely Linear Estimation Filters
   - Definition
   - Why widely-linear?
3. Linear and Augmented Complex Kernel LMS Algorithms
   - Complex Kernel LMS (CKLMS)
   - Augmented Complex Kernel LMS (ACKLMS)
4. Experiments
   - Soft non linear channel
   - Hard non linear channel
5. Future Research

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1. Development of widely-linear kernel-based adaptive filters
2. Determining why widely-linear estimation is better than traditional linear estimation.

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- In A typical linear complex filter the output at time instance n is estimated as

$$\hat{d}(n) = \boldsymbol{w}^H \boldsymbol{z}(n).$$

- In widely-linear filters, we estimate the filter's output as:

$$\tilde{d}(n) = \boldsymbol{w}^H \boldsymbol{z}(n) + \boldsymbol{v}^H \boldsymbol{z}^*(n).$$

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Equivalent notations

If we set $\boldsymbol{w} = \boldsymbol{w}_r + i\boldsymbol{w}_i$, $\boldsymbol{v} = \boldsymbol{v}_r + i\boldsymbol{v}_i$ and $\boldsymbol{z}(n) = \boldsymbol{x}(n) + i\boldsymbol{y}(n)$ we take that:

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$$\hat{d}(n) = \boldsymbol{w}_r^T\boldsymbol{x}(n) + \boldsymbol{w}_i^T\boldsymbol{y}(n) + i\left(\boldsymbol{w}_r^T\boldsymbol{y}(n) - \boldsymbol{w}_i^T\boldsymbol{x}(n)\right)$$

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$$\begin{aligned}\tilde{d}(n) = &\left((\boldsymbol{w}_r^T + \boldsymbol{v}_r^T)\boldsymbol{x} + (\boldsymbol{w}_i^T - \boldsymbol{v}_i^T)\boldsymbol{y}\right)\\ &+ i\left(-(\boldsymbol{w}_i^T + \boldsymbol{v}_i^T)\boldsymbol{x} + (\boldsymbol{w}_r^T - \boldsymbol{v}_r^T)\boldsymbol{y})\right).\end{aligned}$$

The DRC approach

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Given two real input vectors $\boldsymbol{x}(n)$ and $\boldsymbol{y}(n)$ compute linear filters in order to estimate two real processes, $d_r(n)$ and $d_i(n)$, in an optimal way that jointly cares for both $d_r(n)$ and $d_i(n)$

The DRC approach

We express the problem in its multichannel formulation:

$$\begin{pmatrix} \bar{d}_r(n) \\ \bar{d}_i(n) \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_{1,1}^T & \boldsymbol{u}_{1,2}^T \\ \boldsymbol{u}_{2,1}^T & \boldsymbol{u}_{2,2}^T \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{x}(n) \\ \boldsymbol{y}(n) \end{pmatrix} \equiv \boldsymbol{U} \cdot \begin{pmatrix} \boldsymbol{x}(n) \\ \boldsymbol{y}(n) \end{pmatrix}.$$

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In complex notation the DRC formulation gives:

$$\bar{d}(n) = (\boldsymbol{u}_{1,1}^T \boldsymbol{x} + \boldsymbol{u}_{1,2}^T \boldsymbol{y}) + i(\boldsymbol{u}_{2,1}^T \boldsymbol{x} + \boldsymbol{u}_{2,2}^T \boldsymbol{y}).$$

Different Formulations

Let's Summarize:

Linear filter: $$\hat{d}(n) = \boldsymbol{w}_r^T \boldsymbol{x}(n) + \boldsymbol{w}_i^T \boldsymbol{y}(n) + i\left(\boldsymbol{w}_r^T \boldsymbol{y}(n) - \boldsymbol{w}_i^T \boldsymbol{x}(n)\right).$$

Widely-linear filter: $$\tilde{d}(n) = \left((\boldsymbol{w}_r^T + \boldsymbol{v}_r^T)\boldsymbol{x} + (\boldsymbol{w}_i^T - \boldsymbol{v}_i^T)\boldsymbol{y}\right) + i\left(-(\boldsymbol{w}_i^T + \boldsymbol{v}_i^T)\boldsymbol{x} + (\boldsymbol{w}_r^T - \boldsymbol{v}_r^T)\boldsymbol{y}\right).$$

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2. The same is true for the Widely-linear filter.
3. It is not difficult to prove that the DRC and the widely-linear estimation functions are equivalent.

A useful Proposition

In general we can prove the following.

Proposition

Consider a real Hilbert space $\mathcal{H}$, the real Hilbert space $\mathcal{H}^2$ and the complex Hilbert space $\mathbb{H} = \mathcal{H} + i\mathcal{H}$. Then any linear function $\boldsymbol{T} : \mathcal{H}^2 \to \mathbb{R}^2$ can be expressed in complex notation as

$$\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{T}(\boldsymbol{x} + i\boldsymbol{y}) = \boldsymbol{T}(\boldsymbol{z}) = \langle \boldsymbol{z}, \boldsymbol{w} \rangle_{\mathbb{H}} + \langle \boldsymbol{z}^*, \boldsymbol{v} \rangle_{\mathbb{H}},$$

for some $\boldsymbol{w}, \boldsymbol{v} \in \mathbb{H}$, where $\langle \cdot, \cdot, \rangle_{\mathbb{H}}$ is the respective inner product of $\mathbb{H}$.

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The linearity with respect to the field of complex numbers is restricted, compared to the linearity that underlies the DRC approach, which is more natural. Thus, the correct complex linear estimation is $\boldsymbol{T}(\boldsymbol{z}) = \langle \boldsymbol{z}, \boldsymbol{w} \rangle_{\mathbb{H}} + \langle \boldsymbol{z}^*, \boldsymbol{v} \rangle_{\mathbb{H}}$ rather than $\boldsymbol{T}(\boldsymbol{z}) = \langle \boldsymbol{z}, \boldsymbol{w} \rangle_{\mathbb{H}}$.

Outline

Complex Kernel LMS (CKLMS)

Consider the sequence of examples

$$(\boldsymbol{z}(1), d(1)), (\boldsymbol{z}(2), d(2)), \dots, (\boldsymbol{z}(N), d(N)),$$

where $d(n) \in \mathbb{C}$, $\boldsymbol{z}(n) \in \mathbb{C}^{\nu}$,
$\boldsymbol{z}(n) = \boldsymbol{x}(n) + i\boldsymbol{y}(n)$,
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Consider, also, a real RKHS $\mathcal{H}$, the real Hilbert space $\mathcal{H}^2$ and the complex Hilbert space $\mathbb{H} = \mathcal{H} + i\mathcal{H}$.

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The objective of the CKLMS is to estimate $\boldsymbol{w}$, so that to minimize $E(\mathcal{L}_n(\boldsymbol{w}))$, at each time instance n, where

$$\mathcal{L}_n(\boldsymbol{w}) = |e(n)|^2 = \left| d(n) - \hat{d}(n) \right|^2.$$

Complex Kernel LMS (CKLMS)

Applying the rules of Wirtinger's Calculus in complex RKHS, we can easily deduce that the gradient step update of the CKLMS is

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and

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Augmented Complex Kernel LMS (ACKLMS)

We model the widely-linear (augmented) filter as:

$$\tilde{d}(n) = \langle \boldsymbol{\Phi}(\boldsymbol{z}(n)), \boldsymbol{w} \rangle + \langle \boldsymbol{\Phi}^*(\boldsymbol{z}(n)), \boldsymbol{v} \rangle.$$

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If we replace the inner products with the respective complex kernel function we take:

$$\tilde{d}(n) =\mu\sum_{k=1}^{n-1} e(k)\kappa_{\mathbb{C}}(\boldsymbol{z}(n),\boldsymbol{z}(k)) + \mu\sum_{k=1}^{n-1} e(k)\kappa_{\mathbb{C}}^*(\boldsymbol{z}(n),\boldsymbol{z}(k)).$$

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Equalization

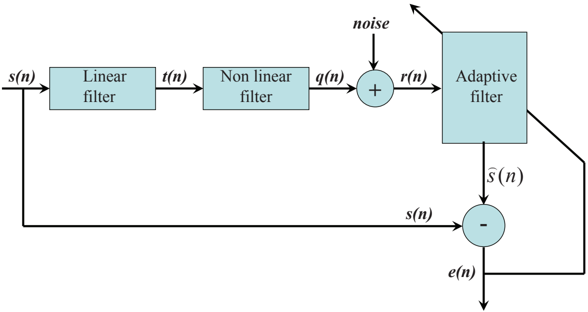

Figure: The equalization problem.

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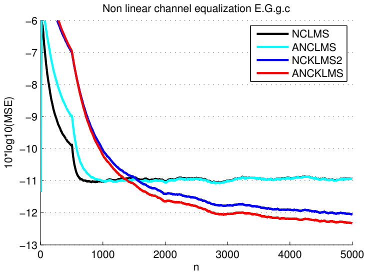
  1. This input is circular for $\rho = \sqrt{2}/2$
  2. highly non-circular if ρ approaches 0 or 1.

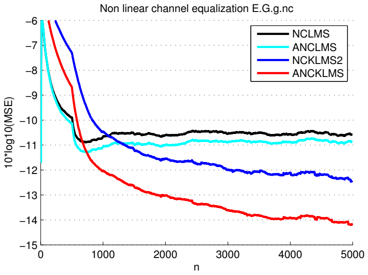
Circular Data

Learning curves for NCLMS, WL-NCLMS, NCKLMS and NACKLMS (filter length $L = 5$, delay $D = 2$) for the soft nonlinear channel equalization, for the **circular** input case.



Non Circular Data

Learning curves for NCLMS, WL-NCLMS, NCKLMS and NACKLMS (filter length $L = 5$, delay $D = 2$) for the soft nonlinear channel equalization, for the **non circular** input case.



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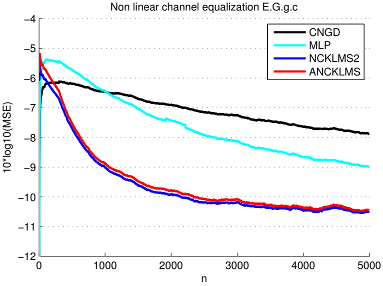
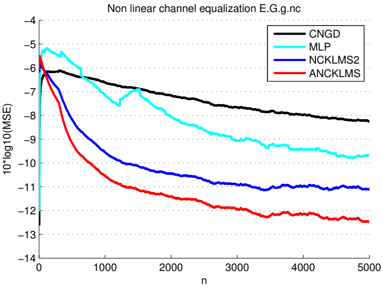
Circular Data

Learning curves for CNGD, MLP (50 nodes), NCKLMS and NACKLMS (filter length $L = 5$, delay $D = 2$) for the hard nonlinear channel equalization, for the **circular** input case.



Non Circular Data

Learning curves for CNGD, MLP (50 nodes), NCKLMS and NACKLMS (filter length $L = 5$, delay $D = 2$) for the hard nonlinear channel equalization, for the **non circular** input case.



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$$C(w, v) = \sum_{k=n-l+1}^{n} \beta_{k-n+l} |e(k)|^2,$$

for a suitable choice of weights $\beta_1, \dots, \beta_l$.

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